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Abstract
Every irreducible finite-dimensional representation of the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$ can be extended to the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_n)$ via the evaluation homomorphism. We give in explicit form the necessary and sufficient conditions for irreducibility of tensor products of such evaluation modules.

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1. Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Finite-dimensional irreducible representations of the corresponding quantum affine algebra $U_q(\hat{\mathfrak{g}})$ were classified by Chari and Pressley [3]; see also [4, chapter 12]. The representations are parametrized by n-tuples of monic polynomials $(P_1(u), \ldots, P_n(u))$, where n is the rank of $\mathfrak{g}$. Moreover, every such representation is isomorphic to a subquotient of a tensor product of the form

$$L_{a_1}(\omega_{i_1}) \otimes \cdots \otimes L_{a_k}(\omega_{i_k}) \tag{1.1}$$

where $L_a(\omega_i)$ denotes the so-called fundamental representation of $U_q(\hat{\mathfrak{g}})$ which corresponds to the n-tuple of polynomials with $P_j(u) = 1$ for all $j \neq i$ and $P_i(u) = u - a$, where $a \in \mathbb{C}$ and ω_i is a fundamental weight. In general, the structure of the tensor product module (1.1) appears to be rather complicated. Only recently, irreducibility conditions for this module were found. These conditions were first conjectured by Akasaka and Kashiwara in [1] and proved there in the cases where $\hat{\mathfrak{g}}$ is of type $A^{(1)}$ or $C^{(1)}$. In some other cases the conjecture was proved in different ways by Frenkel–Mukhin [8], Varagnolo–Vasserot [29] before the general conjecture was settled by Kashiwara [13]. This result was generalized by Chari [2] who gave, in particular, irreducibility conditions for tensor products of the representations corresponding

to n-tuples of polynomials $(P_1(u), \ldots, P_n(u))$ such that $P_j(u) = 1$ for all $j \neq i$ and the roots of $P_i(u)$ form a 'q-string'.

In the case where the Lie algebra $\mathfrak{g}$ is of type A, the corresponding quantum affine algebra admits a class of evaluation modules. Namely, as shown by Jimbo [11], there exists a family of algebra homomorphisms $\mathrm{ev}_a : \mathrm{U}_q(\hat{\mathfrak{g}}) \to \mathrm{U}_q(\mathfrak{g})$ which allows one to extend any finite-dimensional irreducible representation $L(\lambda)$ of $\mathrm{U}_q(\mathfrak{g})$ to a module $L_a(\lambda)$ over $\mathrm{U}_q(\hat{\mathfrak{g}})$, where λ is the highest weight of $L(\lambda)$ and $a \in \mathbb{C}$. In particular, if $\lambda = \omega_i$ is a fundamental weight, the evaluation module coincides with the fundamental representation $L_a(\omega_i)$. Furthermore, any finite-dimensional irreducible representation of $\mathrm{U}_q(\hat{\mathfrak{g}})$ is isomorphic to a subquotient of a tensor product module

$$L_{a_1}(\lambda^{(1)}) \otimes \cdots \otimes L_{a_k}(\lambda^{(k)}) \tag{1.2}$$

see e.g. [4, section 12.2]. Irreducible tensor products of form (1.2) thus provide an explicit realization of a wider class of representations in comparison with modules (1.1). In fact, in the case of $\mathfrak{g} = \mathfrak{sl}_2$ every type 1 irreducible finite-dimensional representation of $\mathrm{U}_q(\widehat{\mathfrak{sl}}_2)$ is isomorphic to a module of form (1.2); see [4, section 12.2].

In this paper, we prove necessary and sufficient conditions for irreducibility of the tensor product (1.2). It is more convenient for us to work with the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ instead of $\mathrm{U}_q(\widehat{\mathfrak{sl}}_n)$. The results can be easily reformulated for the latter algebra as well. Our starting point is the important *binary property* established by Nazarov and Tarasov [24] with the use of an observation made by Kitanine, Maillet and Terras [14, 16]. Namely, representation (1.2) is irreducible if and only if for all $i < j$ the modules $L_{a_i}(\lambda^{(i)}) \otimes L_{a_j}(\lambda^{(j)})$ are irreducible. Therefore, we only need to find an irreducibility criterion for the case of $k = 2$ factors in (1.2). In fact, the binary property is proved in [24] in the Yangian context. An explicit formulation for the quantum affine algebra case can be found in Leclerc, Nazarov and Thibon [15]. A general binary cyclicity property was established by Chari [2] for tensor products of arbitrary irreducible finite-dimensional representations of $\mathrm{U}_q(\hat{\mathfrak{g}})$ with $\mathfrak{g}$ of any type. An irreducibility criterion for the induction products of evaluation modules over the affine Hecke algebras of type A was given in [15]. It implies an irreducibility criterion for the $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$-module $L_a(\lambda) \otimes L_b(\mu)$ where the highest weights satisfy some extra conditions: assuming that λ and μ are partitions (one may do this without loss of generality), one should require that the sum of the lengths of λ and μ does not exceed n. In the case where λ and μ are multiples of fundamental weights, the irreducibility conditions are given by Chari [2].

The same kind of irreducibility questions can be posed for the Yangians $\mathrm{Y}(\mathfrak{g})$; see e.g. [4, section 12.1]. An irreducibility criterion of the tensor product of two arbitrary evaluation modules $L_a(\lambda) \otimes L_b(\mu)$ over the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ is given in [19]. The conditions on λ and μ essentially coincide with those of [15]. Some particular cases of this criterion were also established in [23]. It has been known as a 'folklore theorem' that the finite-dimensional representation theories for the Yangian and the quantum affine algebras are essentially the same. A rigorous result in that direction was recently proved by Varagnolo [28]. It allows one to establish an irreducibility criterion for the quantum affine algebras by using the Yangian criterion of [19]. However, the proof in [28] uses rather involved geometric arguments. The aim of this paper is to give an independent direct proof of the irreducibility criterion appropriately modifying the arguments of [19]. In particular, this requires a development of a q-analogue of the quantum minor techniques employed in [19]. This provides a quantum minor realization of the lowering operators for the quantized algebra $\mathrm{U}_q(\mathfrak{gl}_n)$ and allows a new derivation of the q-analogue of the Gelfand–Tsetlin formulae; cf Jimbo [12], Ueno, Takebayashi and Shibukawa [27], Nazarov and Tarasov [21], Tolstoy [26].

2. Quantized algebras and evaluation modules

Fix a complex parameter q which is nonzero and not a root of unity. Following Jimbo [11], we introduce the q-analogue $U_q(\mathfrak{gl}_n)$ of the universal enveloping algebra $U(\mathfrak{gl}_n)$ as an associative algebra generated by the elements $t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}, e_1, \ldots, e_{n-1}$ and $f_1, \ldots, f_{n-1}$ with the defining relations

$$
\begin{aligned}
& t_i t_j = t_j t_i && t_i t_i^{-1} = t_i^{-1} t_i = 1 \\
& t_i e_j t_i^{-1} = e_j q^{\delta_{ij} - \delta_{i,j+1}} && t_i f_j t_i^{-1} = f_j q^{-\delta_{ij} + \delta_{i,j+1}} \\
& [e_i, f_j] = \frac{k_i - k_i^{-1}}{q - q^{-1}} \delta_{ij} && \text{with} \quad k_i = t_i t_{i+1}^{-1} \\
& [e_i, e_j] = [f_i, f_j] = 0 && \text{if} \quad |i - j| > 1 \\
& [e_i, [e_{i\pm1}, e_i]_q]_q = [f_i, [f_{i\pm1}, f_i]_q]_q = 0
\end{aligned}
\tag{2.1}
$$

where we have used the notation $[a, b]_\zeta = ab - \zeta ba$. The q-analogues of the root vectors are defined inductively by

$$
\begin{aligned}
& e_{i,i+1} = e_i && e_{i+1,i} = f_i \\
& e_{ij} = [e_{ik}, e_{kj}]_q && \text{for} \quad i < k < j \\
& e_{ij} = [e_{ik}, e_{kj}]_{q^{-1}} && \text{for} \quad i > k > j.
\end{aligned}
\tag{2.2}
$$

We shall also use an R-matrix presentation of the algebra $U_q(\mathfrak{gl}_n)$; see e.g. [11] and [25]. Consider the R-matrix

$$
R = q \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i < j} E_{ij} \otimes E_{ji}
\tag{2.3}
$$

which is an element of $\operatorname{End} \mathbb{C}^n \otimes \operatorname{End} \mathbb{C}^n$, where the E_{ij} denote the standard matrix units and the indices run over the set $\{1, \ldots, n\}$. The quantized enveloping algebra $U_q(\mathfrak{gl}_n)$ is generated by the elements t_{ij} and $\bar{t}_{ij}$ with $1 \leqslant i, j \leqslant n$ subject to the relations

$$
\begin{aligned}
& t_{ij} = \bar{t}_{ji} = 0 && 1 \leqslant i < j \leqslant n \\
& t_{ii} \bar{t}_{ii} = \bar{t}_{ii} t_{ii} = 1 && 1 \leqslant i \leqslant n \\
& R T_1 T_2 = T_2 T_1 R && R \bar{T}_1 \bar{T}_2 = \bar{T}_2 \bar{T}_1 R && R \bar{T}_1 T_2 = T_2 \bar{T}_1 R.
\end{aligned}
\tag{2.4}
$$

Here T and $\bar{T}$ are the matrices

$$
T = \sum_{i,j} t_{ij} \otimes E_{ij} \qquad \bar{T} = \sum_{i,j} \bar{t}_{ij} \otimes E_{ij}
\tag{2.5}
$$

which are regarded as elements of the algebra $U_q(\mathfrak{gl}_n) \otimes \operatorname{End} \mathbb{C}^n$. Both sides of each of the R-matrix relations in (2.4) are elements of $U_q(\mathfrak{gl}_n) \otimes \operatorname{End} \mathbb{C}^n \otimes \operatorname{End} \mathbb{C}^n$ and the subscripts of T and $\bar{T}$ indicate the copies of $\operatorname{End} \mathbb{C}^n$ where T or $\bar{T}$ acts; e.g. $T_1 = T \otimes 1$. An isomorphism between the two presentations is given by the formulae

$$
t_i \mapsto t_{ii} \qquad t_i^{-1} \mapsto \bar{t}_{ii} \qquad e_i \mapsto -\frac{\bar{t}_{i,i+1} t_{ii}}{q - q^{-1}} \qquad f_i \mapsto \frac{\bar{t}_{ii} t_{i+1,i}}{q - q^{-1}}.
\tag{2.6}
$$

We shall identify the corresponding elements of $U_q(\mathfrak{gl}_n)$ via this isomorphism.

We now introduce the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_n)$ following [25]; see also [6, 9]. By definition, $U_q(\widehat{\mathfrak{gl}}_n)$ has countably many generators $t_{ij}^{(r)}$ and $\bar{t}_{ij}^{(r)}$ where $1 \leqslant i, j \leqslant n$ and r runs over nonnegative integers. They are combined into the matrices

$$
T(u) = \sum_{i,j=1}^n t_{ij}(u) \otimes E_{ij} \qquad \bar{T}(u) = \sum_{i,j=1}^n \bar{t}_{ij}(u) \otimes E_{ij}
\tag{2.7}
$$

where $t_{ij}(u)$ and $\bar{t}_{ij}(u)$ are formal series in u^{-1} and u, respectively:

$$t_{ij}(u) = \sum_{r=0}^{\infty} t_{ij}^{(r)} u^{-r} \qquad \bar{t}_{ij}(u) = \sum_{r=0}^{\infty} \bar{t}_{ij}^{(r)} u^{r}. \tag{2.8}$$

The defining relations are

$$
\begin{aligned}
&t_{ij}^{(0)} = \bar{t}_{ji}^{(0)} = 0 && 1 \leqslant i < j \leqslant n \\
&t_{ii}^{(0)} \bar{t}_{ii}^{(0)} = \bar{t}_{ii}^{(0)} t_{ii}^{(0)} = 1 && 1 \leqslant i \leqslant n \\
&R(u,v) T_1(u) T_2(v) = T_2(v) T_1(u) R(u,v) \\
&R(u,v) \bar{T}_1(u) \bar{T}_2(v) = \bar{T}_2(v) \bar{T}_1(u) R(u,v) \\
&R(u,v) \bar{T}_1(u) T_2(v) = T_2(v) \bar{T}_1(u) R(u,v).
\end{aligned}
\tag{2.9}
$$

Here $R(u,v)$ is the trigonometric R-matrix given by

$$
\begin{aligned}
R(u,v) = (u-v) \sum_{i \neq j} E_{ii} \otimes E_{jj} &+ (q^{-1}u - qv) \sum_i E_{ii} \otimes E_{ii} \\
&+ (q^{-1} - q) u \sum_{i>j} E_{ij} \otimes E_{ji} + (q^{-1} - q) v \sum_{i<j} E_{ij} \otimes E_{ji}
\end{aligned}
\tag{2.10}
$$

where the subscripts of $T(u)$ and $\bar{T}(u)$ are interpreted in the same way as in (2.4).

Remark 2.1. It is apparently more common to use the name *quantum affine algebra* and notation $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ for the central extension of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$. It is well known, however, that the action of the central element c is trivial in finite-dimensional irreducible representations (see e.g. [4, chapter 12]) so that our terminology should not bring an ambiguity.

The defining relations can easily be rewritten in terms of the generators. In particular, the relations between the $t_{ij}(u)$ take the form

$$
\begin{aligned}
(q^{-\delta_{ik}}u - q^{\delta_{ik}}v) t_{ij}(u) t_{kl}(v) &+ (q^{-1} - q)(u\,\delta_{i>k} + v\,\delta_{i<k}) t_{kj}(u) t_{il}(v) \\
&= (q^{-\delta_{jl}}u - q^{\delta_{jl}}v) t_{kl}(v) t_{ij}(u) + (q^{-1} - q)(u\,\delta_{j<l} + v\,\delta_{j>l}) t_{kj}(v) t_{il}(u)
\end{aligned}
\tag{2.11}
$$

where $\delta_{i>k}$ or $\delta_{i<k}$ is 1 if the inequality for the subscripts holds and 0 otherwise.

A family of the *evaluation homomorphisms* $\mathrm{ev}_a : \mathrm{U}_q(\widehat{\mathfrak{gl}}_n) \to \mathrm{U}_q(\mathfrak{gl}_n)$ is defined by

$$T(u) \mapsto T - a\bar{T}u^{-1} \qquad \bar{T}(u) \mapsto \bar{T} - a^{-1}Tu \tag{2.12}$$

where a is a nonzero complex number. Note that the R-matrix $R(u,v)$ satisfies $R(cu, cv) = c\,R(u,v)$ for any nonzero $c \in \mathbb{C}$. Therefore, the mapping

$$T(u) \mapsto T(cu) \qquad \bar{T}(u) \mapsto \bar{T}(cu) \tag{2.13}$$

defines an automorphism of the algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$. Clearly, the homomorphism ev_a is the composition of such an automorphism with $c = a^{-1}$ and the evaluation homomorphism $\mathrm{ev} = \mathrm{ev}_1$ given by

$$T(u) \mapsto T - \bar{T}u^{-1} \qquad \bar{T}(u) \mapsto \bar{T} - Tu. \tag{2.14}$$

There is a Hopf algebra structure on $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ with the coproduct defined by

$$\Delta(t_{ij}(u)) = \sum_{k=1}^{n} t_{ik}(u) \otimes t_{kj}(u) \qquad \Delta(\bar{t}_{ij}(u)) = \sum_{k=1}^{n} \bar{t}_{ik}(u) \otimes \bar{t}_{kj}(u). \tag{2.15}$$

Finite-dimensional irreducible representations of $\mathrm{U}_q(\mathfrak{gl}_n)$ are completely described by their highest weights; see e.g. [4, chapter 10]. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be an n-tuple of integers

with the condition $\lambda_1 \geqslant \cdots \geqslant \lambda_n$. We shall call such an n-tuple a $\mathfrak{gl}_n$-highest weight. The finite-dimensional irreducible $U_q(\mathfrak{gl}_n)$-module $L(\lambda)$ corresponding to the highest weight λ contains a unique, up to a constant factor, vector ξ such that

$$t_i\xi = q^{\lambda_i}\xi \qquad \text{and} \qquad e_{ij}\xi = 0 \qquad \text{for} \quad i < j. \tag{2.16}$$

Using the evaluation homomorphism (2.12) we can extend $L(\lambda)$ to the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_n)$. We call such an extension the *evaluation module* and denote it by $L_a(\lambda)$. In the case $a = 1$ we keep the notation $L(\lambda)$ for the evaluation module $L_1(\lambda)$. Coproduct (2.15) allows us to form tensor product modules of the type $L_a(\lambda) \otimes L_b(\mu)$ over the algebra $U_q(\widehat{\mathfrak{gl}}_n)$. Our main result is an irreducibility criterion of such modules. We note that without loss of generality both evaluation parameters a and b can be taken to be equal to 1; see section 3 below. In order to formulate the result, we need the following definition [19]; cf [15]. Two disjoint finite subsets A and B of $\mathbb{Z}$ are called *crossing* if there exist elements $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that either $a_1 < b_1 < a_2 < b_2$ or $b_1 < a_1 < b_2 < a_2$. Otherwise, A and B are called *non-crossing*. Given a highest weight λ with integer entries we set $l_i = \lambda_i - i + 1$ and introduce the following subset of $\mathbb{Z}$:

$$\mathcal{A}_\lambda = \{l_1, l_2, \ldots, l_n\}.$$

Theorem 2.2. *The $U_q(\widehat{\mathfrak{gl}}_n)$-module $L(\lambda) \otimes L(\mu)$ is irreducible if and only if the sets $\mathcal{A}_\lambda \setminus \mathcal{A}_\mu$ and $\mathcal{A}_\mu \setminus \mathcal{A}_\lambda$ are non-crossing.*

The proof of this theorem will be given in the following sections. The first step is to reduce the problem to the case of the standard Borel subalgebra of $U_q(\widehat{\mathfrak{gl}}_n)$. We call this subalgebra the q-Yangian and denote by $Y_q(\mathfrak{gl}_n)$; cf [21]. Then we develop an appropriate q-version of the techniques of lowering operators and Gelfand–Tsetlin bases used in [19] for the proof of such criterion in the Yangian case.

3. Evaluation modules over q-Yangians

We define the q-Yangian $Y_q(\mathfrak{gl}_n)$ as the (Hopf) subalgebra of $U_q(\widehat{\mathfrak{gl}}_n)$ generated by the elements $t_{ij}^{(r)}$ with $1 \leqslant i, j \leqslant n$ and $r \geqslant 0$. The restriction of the evaluation homomorphism (2.12) to the q-Yangian is given by the first formula in (2.12), or, equivalently, in terms of the first presentation of $U_q(\mathfrak{gl}_n)$, it can be written as

$$\begin{aligned}
t_{ii}(u) &\mapsto t_i - a\,t_i^{-1}u^{-1} \\
t_{ij}(u) &\mapsto (q - q^{-1})t_j\,e_{ij} && \text{if} \quad i > j \\
t_{ij}(u) &\mapsto a\,(q - q^{-1})e_{ij}\,t_i^{-1}u^{-1} && \text{if} \quad i < j.
\end{aligned} \tag{3.1}$$

The highest weight of an arbitrary finite-dimensional irreducible representation L of $U_q(\mathfrak{gl}_n)$ may have a more general form than (2.16). Namely, if ξ is the highest vector of L then

$$t_i\xi = \alpha_i\xi \qquad \text{and} \qquad e_{ij}\xi = 0 \qquad \text{for} \quad i < j \tag{3.2}$$

for a collection $(\alpha_1, \ldots, \alpha_n)$ of nonzero complex numbers of the form

$$\alpha_i = h\,\varepsilon_i\,q^{\lambda_i} \qquad i = 1, \ldots, n \tag{3.3}$$

where h is a nonzero complex number, each ε_i is equal to 1 or -1, and the λ_i are integers satisfying $\lambda_i \geqslant \lambda_{i+1}$ for all i. We denote the corresponding representation by $L(h, \varepsilon, \lambda)$ where

$$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \qquad \lambda = (\lambda_1, \ldots, \lambda_n). \tag{3.4}$$

The evaluation homomorphism ev_a allows us to regard $L(h, \varepsilon, \lambda)$ as a $Y_q(\mathfrak{gl}_n)$-module which we denote by $L_a(h, \varepsilon, \lambda)$. Using coproduct (2.15) we can consider the tensor products of the form

$$L_a(h, \varepsilon, \lambda) \otimes L_{a'}(h', \varepsilon', \lambda') \tag{3.5}$$

as $Y_q(\mathfrak{gl}_n)$-modules. It is clear that this $Y_q(\mathfrak{gl}_n)$-module coincides with the restriction of (3.5) regarded as a $U_q(\widehat{\mathfrak{gl}}_n)$-module.

Proposition 3.1. *The $U_q(\widehat{\mathfrak{gl}}_n)$-module (3.5) is irreducible if and only if its restriction to the q-Yangian $Y_q(\mathfrak{gl}_n)$ is irreducible.*

Proof. The 'if' part is obviously true. Now suppose, on the contrary, that the $Y_q(\mathfrak{gl}_n)$-module (3.5) contains a nontrivial submodule W. Then, by (2.12) and (2.15), W is invariant with respect to all operators of the form

$$\sum_{k=1}^{n} (t_{ik} - \bar{t}_{ik}u^{-1}a) \otimes (t_{kj} - \bar{t}_{kj}u^{-1}a') \qquad i, j = 1, \ldots, n. \tag{3.6}$$

This implies that W is invariant with respect to the operators

$$\sum_{k=1}^{n} (\bar{t}_{ik} - t_{ik}ua^{-1}) \otimes (\bar{t}_{kj} - t_{kj}ua'^{-1}). \tag{3.7}$$

However, the latter operator is the image of the generator $\bar{t}_{ij}(u)$ of $U_q(\widehat{\mathfrak{gl}}_n)$ in module (3.5). Therefore, W is invariant under the action of the entire algebra $U_q(\widehat{\mathfrak{gl}}_n)$. But this contradicts the irreducibility of (3.5). $\qquad\square$

Remark 3.2. Since every finite-dimensional irreducible representation V of $U_q(\widehat{\mathfrak{gl}}_n)$ is isomorphic to a subquotient of (1.2), the above argument can obviously be extended to show that any such V remains irreducible when restricted to $Y_q(\mathfrak{gl}_n)$.

Proposition 3.1 tells us that the irreducibility conditions for tensor products of evaluation modules over $U_q(\widehat{\mathfrak{gl}}_n)$ and $Y_q(\mathfrak{gl}_n)$ are the same. In what follows we work with $Y_q(\mathfrak{gl}_n)$-modules. Finite-dimensional irreducible representations of $Y_q(\mathfrak{gl}_n)$ can be described in terms of their highest weights in a way similar to the case of the Yangian $Y(\mathfrak{gl}_n)$ [7]; see also [4, chapter 12] and [18]. The highest weight of such a module L is a collection of formal power series $(\nu_1(u), \ldots, \nu_n(u))$ in u^{-1} such that

$$t_{ii}(u)\zeta = \nu_i(u)\zeta \qquad \text{and} \qquad t_{ij}(u)\zeta = 0 \qquad \text{for} \quad i < j \tag{3.8}$$

for a vector $\zeta \in L$ (the highest vector) which is determined uniquely up to a constant factor. If the $Y(\mathfrak{gl}_n)$-module (3.5) is irreducible its highest weight is easy to find from (2.15) and (3.1). It is given by

$$\nu_i(u) = \left(\alpha_i - a\alpha_i^{-1}u^{-1}\right)\left(\alpha_i' - a'\alpha_i'^{-1}u^{-1}\right) \tag{3.9}$$

where the α_i and α_i' are the components of the highest weights of $L_a(h, \varepsilon, \lambda)$ and $L_{a'}(h', \varepsilon', \lambda')$; see (3.3). Note that for a given nondegenerate diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$, the mapping

$$T(u) \mapsto DT(u) \tag{3.10}$$

defines an algebra automorphism of $Y_q(\mathfrak{gl}_n)$, as follows from (2.11). Taking the composition of (3.5) with this automorphism where $d_i = h^{-1}h'^{-1}\varepsilon_i\varepsilon_i'$ we find that the irreducibility of (3.5) is equivalent to that of the module

$$L_b(\lambda) \otimes L_{b'}(\lambda') \qquad b = ah^{-2} \qquad b' = a'h'^{-2} \tag{3.11}$$

where by $L(\lambda)$ we denote the $U_q(\mathfrak{gl}_n)$-module $L(h, \varepsilon, \lambda)$ with $h = 1$ and $\varepsilon = (1, \ldots, 1)$. Similarly, using automorphism (2.13) we find that for any nonzero $c \in \mathbb{C}$ module (3.11) is irreducible if and only if the module $L_{bc}(\lambda) \otimes L_{b'c}(\lambda')$ is. On the other hand, module (3.11) is irreducible unless $b b'^{-1} \in q^{2\mathbb{Z}}$. Analogues of this fact are well known both in the case of Yangians and the quantum affine algebras; cf [4, chapter 12]. One of the ways to prove this is to show that both the module and its dual have no nontrivial singular vectors by considering the eigenvalues of the quantum determinant on the module; see (4.20) below. So, we may now assume that $b' = 1$ and $b = q^{2k}$ in (3.11) for some $k \in \mathbb{Z}$. However, using automorphism (2.13) with $d_i \equiv q^{-k}$ we conclude that the irreducibility of (3.11) is equivalent to that of the module $L_1(\lambda) \otimes L_1(\lambda' - kI)$ where $I = (1, \ldots, 1)$. We shall keep the notation $L(\lambda)$ for the $Y_q(\mathfrak{gl}_n)$-module $L_b(\lambda)$ with $b = 1$. Thus, we may only consider, without loss of generality, the $Y_q(\mathfrak{gl}_n)$-modules of the form $L(\lambda) \otimes L(\mu)$. The irreducibility conditions for a general tensor product module (3.5) can be easily obtained from theorem 2.2.

4. Quantum minor relations

Here we formulate some well-known properties of quantum determinants and quantum minors; see e.g. [5, 21].

Let us consider the multiple tensor product $Y_q(\mathfrak{gl}_n) \otimes (\operatorname{End} \mathbb{C}^n)^{\otimes r}$. We have the following corollary of (2.9) which is verified in the same way as for the Yangians; cf [20]:

$$R(u_1, \ldots, u_r) T_1(u_1) \cdots T_r(u_r) = T_r(u_r) \cdots T_1(u_1) R(u_1, \ldots, u_r) \tag{4.1}$$

where

$$R(u_1, \ldots, u_r) = \prod_{i<j} R_{ij}(u_i, u_j) \tag{4.2}$$

with the product taken in the lexicographical order on the pairs (i, j). Here, like in (2.9), the subscripts of the matrices $T(u)$ and $R(u, v)$ indicate the copies of $\operatorname{End} \mathbb{C}^n$. Consider the q-permutation operator $P \in \operatorname{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ defined by

$$P = \sum_i E_{ii} \otimes E_{ii} + q \sum_{i>j} E_{ij} \otimes E_{ji} + q^{-1} \sum_{i<j} E_{ij} \otimes E_{ji}. \tag{4.3}$$

An action of the symmetric group $\mathfrak{S}_r$ on the space $(\mathbb{C}^n)^{\otimes r}$ can be defined by setting $s_i \mapsto P_{s_i} := P_{i,i+1}$ for $i = 1, \ldots, r-1$, where s_i denotes the transposition $(i, i+1)$. If $\sigma = s_{i_1} \cdots s_{i_l}$ is a reduced decomposition of an element $\sigma \in \mathfrak{S}_r$ we set $P_\sigma = P_{s_{i_1}} \cdots P_{s_{i_l}}$. We denote by A_r the q-antisymmetrizer

$$A_r = \sum_{\sigma \in \mathfrak{S}_r} \operatorname{sgn} \sigma \, P_\sigma. \tag{4.4}$$

The following proposition is proved by induction on r in the same way as for the Yangians [20] with the use of a property of the reduced decompositions ([10], p 50).

Proposition 4.1. *We have the relation in* $\operatorname{End}(\mathbb{C}^n)^{\otimes r}$:

$$R(1, q^{-2}, \ldots, q^{-2r+2}) = \prod_{0 \leqslant i < j \leqslant r-1} (q^{-2i} - q^{-2j}) A_r. \tag{4.5}$$

Now (4.1) implies that

$$A_r T_1(u) \cdots T_r(q^{-2r+2}u) = T_r(q^{-2r+2}u) \cdots T_1(u) A_r \tag{4.6}$$

which equals

$$\sum_{a_i,b_i} t^{a_1\cdots a_r}_{b_1\cdots b_r}(u) \otimes E_{a_1 b_1} \otimes \cdots \otimes E_{a_r b_r} \tag{4.7}$$

for some elements $t^{a_1\cdots a_r}_{b_1\cdots b_r}(u) \in Y_q(\mathfrak{gl}_n)[[u^{-1}]]$ which we call the *quantum minors*. They can be given by the following formulae which are immediate from the definition. If $a_1 < \cdots < a_r$ then

$$t^{a_1\cdots a_r}_{b_1\cdots b_r}(u) = \sum_{\sigma \in \mathfrak{S}_r} (-q)^{-l(\sigma)} \, t_{a_{\sigma(1)} b_1}(u) \cdots t_{a_{\sigma(r)} b_r}(q^{-2r+2}u) \tag{4.8}$$

and for any $\tau \in \mathfrak{S}_r$ we have

$$t^{a_{\tau(1)}\cdots a_{\tau(r)}}_{b_1\cdots b_r}(u) = (-q)^{l(\tau)} t^{a_1\cdots a_r}_{b_1\cdots b_r}(u). \tag{4.9}$$

Here $l(\sigma)$ denotes the length of the permutation σ. If $b_1 < \cdots < b_r$ (and the a_i are arbitrary) then

$$t^{a_1\cdots a_r}_{b_1\cdots b_r}(u) = \sum_{\sigma \in \mathfrak{S}_r} (-q)^{l(\sigma)} \, t_{a_r b_{\sigma(r)}}(q^{-2r+2}u) \cdots t_{a_1 b_{\sigma(1)}}(u) \tag{4.10}$$

and for any $\tau \in \mathfrak{S}_r$ we have

$$t^{a_1\cdots a_r}_{b_{\tau(1)}\cdots b_{\tau(r)}}(u) = (-q)^{-l(\tau)} t^{a_1\cdots a_r}_{b_1\cdots b_r}(u). \tag{4.11}$$

Note also that the quantum minor is zero if two top or two bottom indices are equal.

As another application of (4.1) we obtain the relations between the generators $t_{ij}(u)$ and the quantum minors. For this we introduce an extra copy of End $\mathbb{C}^n$ as a tensor factor which will be enumerated by the index 0. Now specialize the parameters u_i as follows:

$$u_0 = u \qquad u_i = q^{-2i+2}v \qquad \text{for} \quad i = 1, \ldots, r. \tag{4.12}$$

Then by proposition 4.1, element (4.2) will take the form

$$R(u, v, \ldots, q^{-2r+2}v) = \prod_{i=1}^{r} R_{0i}(u, q^{-2i+2}v) A_r. \tag{4.13}$$

Using the definition of the quantum minors (4.7) and equating the matrix elements on both sides of (4.1) we get the required relations. To write them down, let us fix indices $a, b, c_1 < \cdots < c_r$ and $d_1 < \cdots < d_r$. Then we have

$$A_{a,b,(c),(d)}(u, v) = B_{a,b,(c),(d)}(u, v) \tag{4.14}$$

where

$$A_{a,b,(c),(d)}(u, v) = (u - v)t_{ab}(u)t^{c_1\cdots c_r}_{d_1\cdots d_r}(v) + (q^{-1} - q)u \sum_{i=1}^{k} (-q)^{k-i} t_{c_i b}(u) t^{c_1\cdots \hat{c}_i \cdots c_k a c_{k+1}\cdots c_r}_{d_1\cdots d_r}(v)$$

$$+ (q^{-1} - q)v \sum_{i=k+1}^{r} (-q)^{k-i+1} t_{c_i b}(u) t^{c_1\cdots c_k a c_{k+1}\cdots \hat{c}_i\cdots c_r}_{d_1\cdots d_r}(v) \tag{4.15}$$

if $c_k < a < c_{k+1}$ for some $k \in \{0, 1, \ldots, r\}$, and

$$A_{a,b,(c),(d)}(u, v) = (q^{-1}u - q\,v)t_{ab}(u)t^{c_1\cdots c_r}_{d_1\cdots d_r}(v) \tag{4.16}$$

if $a = c_k$ for some k. Furthermore,

$$B_{a,b,(c),(d)}(u, v) = (u - v)t^{c_1\cdots c_r}_{d_1\cdots d_r}(v)t_{ab}(u) + (q^{-1} - q)v \sum_{i=1}^{l} (-q)^{i-l} t^{c_1\cdots c_r}_{d_1\cdots \hat{d}_i\cdots d_l b d_{l+1}\cdots d_r}(v)t_{ad_i}(u)$$

$$+ (q^{-1} - q)u \sum_{i=l+1}^{r} (-q)^{i-l-1} t^{c_1\cdots c_r}_{d_1\cdots d_l b d_{l+1}\cdots \hat{d}_i\cdots d_r}(v)t_{ad_i}(u) \tag{4.17}$$

if $d_l < b < d_{l+1}$ for some $l \in \{0, 1, \ldots, r\}$, and

$$B_{a,b,(c),(d)}(u, v) = (q^{-1}u - q\,v)t_{d_1\cdots d_r}^{c_1\cdots c_r}(v)t_{ab}(u) \tag{4.18}$$

if $b = d_l$ for some l; the hats indicate that the indices are omitted. In particular, (4.14) implies the well-known property of the quantum minors: for any indices i, j we have

$$\left[t_{c_i d_j}(u), t_{d_1\cdots d_r}^{c_1\cdots c_r}(v)\right] = 0. \tag{4.19}$$

This implies that all the coefficients of the series

$$\mathrm{q}\det T(u) = t_{1\cdots n}^{1\cdots n}(u) \tag{4.20}$$

belong to the centre of $Y_q(\mathfrak{gl}_n)$. Element (4.20) is called the *quantum determinant* of the matrix $T(u)$. The *quantum comatrix* $\hat{T}(u)$ is defined by the relation

$$\hat{T}(u)T(q^{-2n+2}u) = \mathrm{q}\det T(u). \tag{4.21}$$

Using definition (4.7) we find that $\hat{T}(u) = \sum_{i,j} \hat{t}_{ij}(u) \otimes E_{ij}$ where

$$\hat{t}_{ij}(u) = (-q)^{j-i} t_{1\cdots\hat{i}\cdots n}^{1\cdots\hat{j}\cdots n}(u). \tag{4.22}$$

The elements $\hat{t}_{ij}(u)$ satisfy quadratic relations which can be written in the R-matrix form

$$R(u, v)\hat{T}_2(v)\hat{T}_1(u) = \hat{T}_1(u)\hat{T}_2(v)R(u, v). \tag{4.23}$$

To see this, we multiply both sides of the first R-matrix relation in (2.9) by the product $T_1(u)^{-1}T_2(v)^{-1}$ from the left and by $T_2(v)^{-1}T_1(u)^{-1}$ from the right. Then substitute $u \mapsto q^{-2n+2}u$, $v \mapsto q^{-2n+2}v$ and multiply both sides by $\mathrm{q}\det T(u)\,\mathrm{q}\det T(v)$ to get (4.23). It is easy to rewrite this relation in terms of $\hat{t}_{ij}(u)$ in a way similar to (2.11). We shall also need the following relation between the quantum minors.

Proposition 4.2. *We have*

$$t_{1\cdots n-2,n}^{2\cdots n}(u)t_{2\cdots n-1}^{2\cdots n-1}(u) = t_{1\cdots n-2}^{2\cdots n-1}(u)t_{2\cdots n}^{2\cdots n}(u) + qt_{1\cdots n-1}^{2\cdots n}(u)t_{2\cdots n-2,n}^{2\cdots n-1}(u). \tag{4.24}$$

Proof. We find from the definition of the quantum determinant that

$$A_n T_1(u)\cdots T_{n-2}(q^{-2n+6}u) = \mathrm{q}\det T(u) A_n T_n(q^{-2n+2}u)^{-1}T_{n-1}(q^{-2n+4}u)^{-1}. \tag{4.25}$$

Equating the matrix elements of both sides and using (4.21) we arrive at the following relation: for $i < j$ and $k < l$,

$$(-q)^{i+j-k-l}t_{1\cdots\hat{k}\cdots\hat{l}\cdots n}^{1\cdots\hat{i}\cdots\hat{j}\cdots n}(u)\,\mathrm{q}\det T(q^2u) = \hat{t}_{lj}(u)\hat{t}_{ki}(q^2u) - q^{-1}\hat{t}_{kj}(u)\hat{t}_{li}(q^2u). \tag{4.26}$$

The right-hand side is a 2×2-quantum minor of the matrix $\hat{T}(u)$ and we denote it by $\hat{t}_{ij}^{kl}(u)$. Furthermore, by analogy with (4.1) we obtain from (4.23)

$$R(u_1, u_2, u_3)\hat{T}_3(u_3)\hat{T}_2(u_2)\hat{T}_1(u_1) = \hat{T}_1(u_1)\hat{T}_2(u_2)\hat{T}_3(u_3)R(u_1, u_2, u_3). \tag{4.27}$$

Now specialize $u_1 = q^2v$, $u_2 = v$, $u_3 = u$ and take the coefficients at $E_{n-1,1} \otimes E_{nn} \otimes E_{11}$ on both sides. Using proposition 4.1 and (4.26) we get

$$(q^{-1} - q)v\hat{t}_{n-1,1}(u)\hat{t}_{1n}^{1n}(v) - (1 - q^2)v\hat{t}_{n1}(u)\hat{t}_{1n}^{1,n-1}(v) + (v - u)\hat{t}_{11}(u)\hat{t}_{1n}^{n-1,n}(v)$$
$$= (q^{-1}v - q\,u)\hat{t}_{1n}^{n-1,n}(v)\hat{t}_{11}(u). \tag{4.28}$$

By putting $u = v$ and rewriting this relation in terms of quantum minors we come to (4.24). $\qquad\square$

The following proposition is proved in the same way as its Yangian counterpart [22]; see also [20].

Proposition 4.3. *The images of the quantum minors under the coproduct are given by*

$$\Delta\left(t^{a_1\cdots a_r}_{b_1\cdots b_r}(u)\right) = \sum_{c_1<\cdots<c_r} t^{a_1\cdots a_r}_{c_1\cdots c_r}(u) \otimes t^{c_1\cdots c_r}_{b_1\cdots b_r}(u)$$

where the summation is over all subsets of indices $\{c_1, \ldots, c_r\}$ *from* $\{1, \ldots, n\}$.

5. Gelfand–Tsetlin basis in $L(\lambda)$

It has been observed in [17] that the raising and lowering operators for the reduction $\mathfrak{gl}_n \downarrow \mathfrak{gl}_{n-1}$ (and more generally for the corresponding Yangian reduction) can be given by quantum minor formulae. This was used to construct analogues of the Gelfand–Tsetlin basis for generic Yangian modules. Here we modify the arguments of [17] to give q-analogues of the quantum minor formulae and construct a basis of Gelfand–Tsetlin-type for the $U_q(\mathfrak{gl}_n)$-module $L(\lambda)$. Some other constructions of such bases can be found in [12, 21, 26, 27].

A *pattern* Λ (associated with λ) is a sequence of rows of integers $\Lambda_n, \Lambda_{n-1}, \ldots, \Lambda_1$, where $\Lambda_r = (\lambda_{r1}, \ldots, \lambda_{rr})$ is the rth row from the bottom, the top row Λ_n coincides with λ, and the following *betweenness conditions* are satisfied: for $r = 1, \ldots, n-1$

$$\lambda_{r+1,i+1} \leqslant \lambda_{ri} \leqslant \lambda_{r+1,i} \qquad \text{for} \quad i = 1, \ldots, r. \tag{5.1}$$

We shall be using the notation $l_{ki} = \lambda_{ki} - i + 1$. Also, for any integer m we set

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}. \tag{5.2}$$

Proposition 5.1. *There exists a basis* $\{\xi_\Lambda\}$ *in* $L(\lambda)$ *parametrized by the patterns* Λ *such that the action of the generators of* $U_q(\mathfrak{gl}_n)$ *is given by*

$$t_k\xi_\Lambda = q^{w_k}\xi_\Lambda \qquad w_k = \sum_{i=1}^{k}\lambda_{ki} - \sum_{i=1}^{k-1}\lambda_{k-1,i} \tag{5.3}$$

$$e_k\xi_\Lambda = -\sum_{j=1}^{k}\frac{[l_{k+1,1}-l_{kj}]\cdots[l_{k+1,k+1}-l_{kj}]}{[l_{k1}-l_{kj}]\cdots\wedge_j\cdots[l_{kk}-l_{kj}]}\xi_{\Lambda+\delta_{kj}} \tag{5.4}$$

$$f_k\xi_\Lambda = \sum_{j=1}^{k}\frac{[l_{k-1,1}-l_{kj}]\cdots[l_{k-1,k-1}-l_{kj}]}{[l_{k1}-l_{kj}]\cdots\wedge_j\cdots[l_{kk}-l_{kj}]}\xi_{\Lambda-\delta_{kj}} \tag{5.5}$$

where $\Lambda \pm \delta_{kj}$ *is obtained from* Λ *by replacing the entry* λ_{kj} *with* $\lambda_{kj} \pm 1$, *and* ξ_Λ *is supposed to be equal to zero if* Λ *is not a pattern; the symbol* $\wedge_j$ *indicates that the jth factor is skipped.*

Proof. Our proof of this result is based on the relations between the quantum minors given in section 4. Set

$$T_{ij}(u) = \frac{ut_{ij} - u^{-1}\bar{t}_{ij}}{q - q^{-1}}. \tag{5.6}$$

Clearly, $(q - q^{-1})T_{ij}(u) = u \operatorname{ev}(t_{ij}(u^2))$; see (2.14). We also define the corresponding quantum minors $T^{a_1\cdots a_r}_{b_1\cdots b_r}(u)$ by formula (4.8) or (4.10) where all series $t_{ij}(u)$ are respectively replaced by $T_{ij}(u)$. Now for any $1 \leqslant a < r \leqslant n$ introduce the *lowering operators* by

$$\tau_{ra}(u) = q^{r-a}T^{a+1\cdots r}_{a\cdots r-1}(u). \tag{5.7}$$

Note that by (4.19) we have

$$\tau_{ra}(u)\tau_{sb}(v) = \tau_{sb}(v)\tau_{ra}(u) \tag{5.8}$$

if $b \leqslant a$ and $s \geqslant r$.

Let $\mu = (\mu_1, \ldots, \mu_{n-1})$ be an $(n-1)$-tuple of integers satisfying the inequalities

$$\lambda_{i+1} \leqslant \mu_i \leqslant \lambda_i \qquad i = 1, \ldots, n-1. \tag{5.9}$$

Introduce the vector

$$\xi_\mu = \prod_{a=1}^{n-1} \tau_{na}(q^{-\mu_a - 1}) \cdots \tau_{na}(q^{-\lambda_a + 1}) \tau_{na}(q^{-\lambda_a}) \xi \tag{5.10}$$

where ξ is the highest vector of $L(\lambda)$. An easy induction with the use of (4.14) shows that ξ_μ satisfies

$$e_k \xi_\mu = 0 \qquad k = 1, \ldots, n-2 \qquad \text{and} \qquad t_k \xi_\mu = q^{\mu_k} \xi_\mu \qquad k = 1, \ldots, n-1. \tag{5.11}$$

Thus, if ξ_μ is nonzero then it generates a $U_q(\mathfrak{gl}_{n-1})$-submodule isomorphic to $L(\mu)$. Now given a pattern Λ, we define vectors $\xi_\Lambda \in L(\lambda)$ by

$$\xi_\Lambda = \overset{\rightarrow}{\prod_{r=2,\ldots,n}} \prod_{a=1}^{r-1} \tau_{ra}(q^{-\lambda_{r-1,a} - 1}) \cdots \tau_{ra}(q^{-\lambda_{ra} + 1}) \tau_{ra}(q^{-\lambda_{ra}}) \xi. \tag{5.12}$$

Relation (5.3) easily follows from (5.11) and the defining relations in $U_q(\mathfrak{gl}_n)$. We now derive formulae (5.4) and (5.5) which together with (5.3) will imply that the vectors ξ_Λ are linearly independent and form a nontrivial submodule of $L(\lambda)$. Since $L(\lambda)$ is irreducible this submodule must coincide with $L(\lambda)$. Below we shall only give a derivation of (5.4); the proof of (5.5) is quite similar and will be omitted. Note first, that, as follows from (4.14), if $k \geqslant r$ then e_k commutes with the lowering operator $\tau_{ra}(u)$. Therefore, we only need to apply e_{n-1} to the vector ξ_μ defined in (5.10). Let $a \in \{1, \ldots, n-1\}$ be the least index such that $\lambda_a - \mu_a > 0$. We use a reverse induction on a with the trivial base $a = n$ (i.e. $\xi_\mu = \xi$). For a nonnegative integer m, we introduce the products of the lowering operators by

$$\mathcal{T}_{na}(u, m) = \prod_{i=1}^{m} \tau_{na}(q^{i-1} u). \tag{5.13}$$

We then have

$$\xi_\mu = \mathcal{T}_{na}(q^{-\lambda_a}, \lambda_a - \mu_a) \xi_{\mu'} \tag{5.14}$$

where μ' is obtained from μ by replacing μ_a with λ_a. We derive from (4.14) that

$$e_{n-1} \tau_{na}(u) = q^{-1} \tau_{na}(u) e_{n-1} - q^{n-a-1} T_{a\cdots n-2,n}^{a+1\cdots n}(u). \tag{5.15}$$

Then by induction we obtain

$$e_{n-1} \mathcal{T}_{na}(u, m) = q^{-m} \mathcal{T}_{na}(u, m) e_{n-1} - \sum_{i=1}^{m} q^{n-a-i} \tau_{na}(u) \cdots T_{a\cdots n-2,n}^{a+1\cdots n}(q^{i-1} u) \cdots \tau_{na}(q^{m-1} u). \tag{5.16}$$

Consider now the subalgebra Y_a of $\mathrm{Y}_q(\mathfrak{gl}_n)$ generated by the coefficients of $t_{ij}(u)$ with $a \leqslant i, j \leqslant n$. Using relations (4.23) for this subalgebra we get

$$T_{a\cdots n-2,n}^{a+1\cdots n}(u) \tau_{na}(qu) = q \tau_{na}(u) T_{a\cdots n-2,n}^{a+1\cdots n}(qu). \tag{5.17}$$

This brings (5.16) to the form

$$e_{n-1}\mathcal{T}_{na}(u,m) = q^{-m}\mathcal{T}_{na}(u,m)e_{n-1} - [m]q^{n-a-1}\mathcal{T}_{na}(u,m-1)T_{a\cdots n-2,n}^{a+1\cdots n}(q^{m-1}u). \qquad (5.18)$$

By the induction hypothesis, the action of e_{n-1} on $\xi_{\mu'}$ is found from (5.4). So we only need to calculate $T_{a\cdots n-2,n}^{a+1\cdots n}(u)\xi_{\mu'}$ at $u = q^{-\mu_a-1}$. For this we use proposition 4.2. Clearly, relation (4.24) remains valid if we replace each quantum minor with the corresponding minor in the elements $T_{ij}(u)$. By the induction hypothesis, we have

$$T_{a+1\cdots n-1}^{a+1\cdots n-1}(q^{-\mu_a-1})\xi_{\mu'} = \prod_{i=a+1}^{n-1} [m_i - m_a]\xi_{\mu'} \qquad (5.19)$$

where $m_i = \mu_i - i + 1$. Similarly, since $T_{a+1\cdots n}^{a+1\cdots n}(u)$ commutes with the lowering operators $\tau_{nb}(v)$, we obtain

$$T_{a+1\cdots n}^{a+1\cdots n}(q^{-\mu_a-1})\xi_{\mu'} = \prod_{i=a+1}^{n} [l_i - m_a]\xi_{\mu'}. \qquad (5.20)$$

Moreover, by (4.14) we have

$$T_{a+1\cdots n-2,n}^{a+1\cdots n-1}(u) = \left[T_{a+1\cdots n-1}^{a+1\cdots n-1}(u), e_{n-1}\right]_q \qquad (5.21)$$

and so, the action of $T_{a+1\cdots n-2,n}^{a+1\cdots n-1}(q^{-\mu_a-1})$ on $\xi_{\mu'}$ is also found by induction with the use of (5.19). It is now a matter of a straightforward calculation to check that the resulting expression for the matrix elements of e_{n-1} agrees with (5.4). $\qquad\qquad\square$

6. Proof of theorem 2.2

Here we outline the main arguments in the proof of theorem 2.2. They closely follow the proof of its Yangian version of [19] with the use of the quantum minor relations given in section 4. For a pair of indices $i < j$ we shall denote

$$\begin{aligned}
\langle l_j, l_i \rangle &= \{l_j, l_j + 1, \ldots, l_i\} \backslash \{l_j, l_{j-1}, \ldots, l_i\} \\
\langle m_j, m_i \rangle &= \{m_j, m_j + 1, \ldots, m_i\} \backslash \{m_j, m_{j-1}, \ldots, m_i\}
\end{aligned} \qquad (6.1)$$

where $l_i = \lambda_i - i + 1$ and $m_i = \mu_i - i + 1$. In particular, if $\lambda_i = \lambda_{i+1} = \cdots = \lambda_j$ then $\langle l_j, l_i \rangle = \emptyset$. It was shown in [19, proposition 2.8] that the condition of theorem 2.2 is equivalent to the following: for all pairs of indices $1 \leqslant i < j \leqslant n$ we have

$$m_j, m_i \notin \langle l_j, l_i \rangle \qquad \text{or} \qquad l_j, l_i \notin \langle m_j, m_i \rangle. \qquad (6.2)$$

We start by proving that these conditions are sufficient for the irreducibility of the $Y_q(\mathfrak{gl}_n)$-module $L(\lambda) \otimes L(\mu)$. Let ξ and ξ' denote the highest vectors of the $\mathfrak{gl}_n$-modules $L(\lambda)$ and $L(\mu)$, respectively. The key part of the proof of sufficiency of the conditions is to show by induction on n that if $\zeta \in L(\lambda) \otimes L(\mu)$ is a nonzero vector satisfying (3.8) for some series $\nu_i(u)$ then

$$\zeta = \text{const}\, \xi \otimes \xi'. \qquad (6.3)$$

Then considering dual modules we also show that the vector $\xi \otimes \xi'$ is cyclic.

Note that the modules $L(\lambda) \otimes L(\mu)$ and $L(\mu) \otimes L(\lambda)$ are simultaneously reducible or irreducible. This can be easily deduced from formula (3.9) for the highest weight of the irreducible module (3.5). So, we may assume without loss of generality that

$$m_1, m_n \notin \langle l_n, l_1 \rangle. \qquad (6.4)$$

Consider the Gelfand–Tsetlin basis $\{\xi_\Lambda\}$ of the $U_q(\mathfrak{gl}_n)$-module $L(\lambda)$; see section 5. The singular vector ζ is uniquely written in the form

$$\zeta = \sum_\Lambda \xi_\Lambda \otimes \eta_\Lambda \tag{6.5}$$

summed over all patterns Λ associated with λ, and $\eta_\Lambda \in L(\mu)$. We define the weight of a pattern Λ as the n-tuple $w(\Lambda) = (w_1, \dots, w_n)$ where the w_k are given in (5.3). We use a standard partial ordering on the weights such that $w \preceq w'$ if and only if $w' - w$ is a $\mathbb{Z}_+$-linear combination of the elements $\varepsilon_i - \varepsilon_{i+1}$, where ε_i is the n-tuple with 1 on the ith place and zeros elsewhere. Choose a minimal pattern Λ^0 with respect to this ordering among those occurring in expansion (6.5). Then, exactly as in [19, lemmas 3.2 and 3.3], we show that η_{Λ^0} is proportional to the highest vector ξ' of $L(\mu)$ and that Λ^0 is determined uniquely. Furthermore, we apply proposition 5.1 to demonstrate that due to condition (6.4), the $(n-1)$th row of Λ^0 is $\lambda_- := (\lambda_1, \dots, \lambda_{n-1})$. This means that vector (6.5) belongs to the $Y_q(\mathfrak{gl}_{n-1})$-span of the vector $\xi \otimes \xi'$, which is isomorphic to the tensor product $L(\lambda_-) \otimes L(\mu_-)$. Since conditions (6.2) hold for λ_- and μ_-, we conclude by induction that (6.3) holds.

The next step is to show that under conditions (6.2) (assuming (6.4) as well) the vector $\xi \otimes \xi'$ of the $Y_q(\mathfrak{gl}_n)$-module $L = L(\lambda) \otimes L(\mu)$ is cyclic. The cyclicity of L is equivalent to the cocyclicity of the dual module $L^* = L(\lambda)^* \otimes L(\mu)^*$ (that is, to the fact that any singular vector of L^* is proportional to $\xi^* \otimes \xi'^*$). To define the dual modules $L(\lambda)^*$ and $L(\mu)^*$ we use the anti-automorphism σ of $U_q(\mathfrak{gl}_n)$ defined by

$$\sigma : e_i \mapsto -e_i \qquad \sigma : f_i \mapsto -f_i \qquad \sigma : t_i \mapsto t_i^{-1}. \tag{6.6}$$

The dual space $L(\lambda)^*$ becomes a $U_q(\mathfrak{gl}_n)$-module if we set

$$(yf)(v) = f(\sigma(y)v) \qquad y \in U_q(\mathfrak{gl}_n) \qquad f \in L(\lambda)^* \qquad v \in L(\lambda). \tag{6.7}$$

It is easy to see that the $U_q(\mathfrak{gl}_n)$-module $L(\lambda)^*$ is isomorphic to $L(-\lambda^\omega)$, where $\lambda^\omega = (\lambda_n, \dots, \lambda_1)$, and so

$$L^* \simeq L(-\lambda^\omega) \otimes L(-\mu^\omega). \tag{6.8}$$

Next we verify that if N is any submodule of L then its annihilator

$$\operatorname{Ann} N = \{f \in L^* \mid f(v) = 0 \quad \text{for all} \quad v \in N\}$$

is a nonzero submodule in L^*. The claim now follows from the fact that if ζ' is a lowest singular vector of module (6.8) then ζ' is proportional to $\eta \otimes \eta'$, where η and η' are the lowest vectors of $L(\lambda)$ and $L(\mu)$, respectively. This is proved by repeating the above argument for the singular vector ζ.

To prove the necessity of the conditions of the theorem we use induction on n again. It is not difficult to see that if the $Y_q(\mathfrak{gl}_n)$-module $L(\lambda) \otimes L(\mu)$ is irreducible then so are the $Y_q(\mathfrak{gl}_{n-1})$-modules $L(\lambda_1, \dots, \lambda_{n-1}) \otimes L(\mu_1, \dots, \mu_{n-1})$ and $L(\lambda_2, \dots, \lambda_n) \otimes L(\mu_2, \dots, \mu_n)$. Therefore, by the induction hypothesis, conditions (6.2) can only be violated for $i = 1$ and $j = n$. Suppose this is the case. Swapping λ and μ if necessary, we may assume that $m_n \in \langle l_n, l_1 \rangle$ and $l_1 \in \langle m_n, m_1 \rangle$. There are two cases. First,

$$m_n \in \langle l_n, l_{n-1} \rangle \qquad \text{and} \qquad l_1 \in \langle m_2, m_1 \rangle. \tag{6.9}$$

Then there exist indices r and s such that

$$m_2, \dots, m_r \in \{l_2, \dots, l_s\} \qquad l_{s+1}, \dots, l_{n-1} \in \{m_{r+1}, \dots, m_{n-1}\}.$$

In particular, this implies that

$$l_i - m_i \in \mathbb{Z}_+ \qquad \text{for all} \quad i = 2, \dots, n-1. \tag{6.10}$$

The second case is

$$m_n \in \langle l_{p+1}, l_p \rangle \qquad \text{and} \qquad l_1 \in \langle m_{n-p+1}, m_{n-p} \rangle \qquad (6.11)$$

for some $1 \leqslant p \leqslant n - 2$. Then $l_{p-i+1} = m_{n-i}$ for $i = 1, \ldots, p - 1$.

We show that in both cases the module $L(\lambda) \otimes L(\mu)$ contains a singular vector which is not proportional to $\xi \otimes \xi'$. Indeed, recall that $t_{ij}(u)$ acts in the tensor product as operator (3.6) with $a = a' = 1$. Therefore, the operator $T_{ij}(u) = u^2 t_{ij}(u^2)$ in $L(\lambda) \otimes L(\mu)$ is polynomial in u. By analogy with (5.7), we introduce the lowering operators

$$\tau_{ra}(u) = T_{a \cdots r-1}^{a+1 \cdots r}(u) \qquad (6.12)$$

and their products

$$\mathcal{T}_{na}(u, k) = \prod_{i=1}^{k} \tau_{na}(q^{i-1} u) \qquad (6.13)$$

where k is a nonnegative integer. The numbers

$$k_i = l_i - m_{n-p+i} \qquad i = 1, \ldots, p$$

are positive integers and we define the vector $\theta \in L(\lambda) \otimes L(\mu)$ by

$$\theta = \mathcal{T}_{n-p+1,1}(q^{-\lambda_1}, k_1) \mathcal{T}'_{n-p+2,2}(q^{-\lambda_2}, k_2) \cdots \mathcal{T}'_{np}(q^{-\lambda_p}, k_p)(\xi \otimes \xi')$$

where $\mathcal{T}'_{na}(u, k)$ is the derivative of the polynomial $\mathcal{T}_{na}(u, k)$. We need to prove that θ is annihilated by all operators $T_{ij}(u)$ with $i < j$. It suffices to show that

$$T_{1 \cdots k-1, k+1}^{1 \cdots k}(u) \theta = 0 \qquad k = 1, \ldots, n-1. \qquad (6.14)$$

Note that by (4.14) we have

$$T_{1 \cdots k-1, k+1}^{1 \cdots k}(u) = \left[T_{1 \cdots k}^{1 \cdots k}(u), e_k \right]_q. \qquad (6.15)$$

Since the element $t_{1 \cdots k}^{1 \cdots k}(u)$ is central in the subalgebra of $Y_q(\mathfrak{gl}_n)$ generated by $t_{ij}(u)$ with $1 \leqslant i, j \leqslant k$ the calculation is essentially reduced to showing that $e_k \theta = 0$ for all k. The action of e_k on θ is found by a modified version of the argument which we used in the derivation of (5.4); cf [19, lemma 4.6].

Finally, to prove that $\theta \neq 0$ we write it as a linear combination

$$\theta = \sum_{\Lambda, M} c_{\Lambda, M} \xi_\Lambda \otimes \xi'_M \qquad (6.16)$$

where ξ_Λ and ξ'_M are the Gelfand–Tsetlin basis vectors in $L(\lambda)$ and $L(\mu)$. Applying proposition 4.3 we show that (6.16) has the form

$$\theta = c \xi_\Lambda \otimes \xi' + \cdots \qquad (6.17)$$

where c is a nonzero constant and

$$\xi_\Lambda = \mathcal{T}_{n-p+1,1}(q^{-\lambda_1}, k_1) \mathcal{T}_{n-p+2,2}(q^{-\lambda_2}, k_2) \cdots \mathcal{T}_{np}(q^{-\lambda_p}, k_p) \xi \qquad (6.18)$$

is a vector of the Gelfand–Tsetlin basis of $L(\lambda)$; see (5.12). Thus, $\theta \neq 0$ which completes the proof of theorem 2.2.

Acknowledgments

References

[1] Akasaka T and Kashiwara M 1997 Finite-dimensional representations of quantum affine algebras *Publ. Res. Inst. Math. Sci.* **33** 839–67

[2] Chari V 2002 Braid group actions and tensor products *Int. Math. Res. Not.* **7** 357–82

[3] Chari V and Pressley A 1991 Quantum affine algebras *Commun. Math. Phys.* **142** 261–83

[4] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)

[5] Cherednik I V 1987 A new interpretation of Gelfand–Tzetlin bases *Duke Math. J.* **54** 563–77

[6] Ding J 1999 Spinor representations of $U_q(\widehat{\mathfrak{gl}}(n))$ and quantum boson–fermion correspondence *Commun. Math. Phys.* **200** 399–420

[7] Drinfeld V G 1988 A new realization of Yangians and quantized affine algebras *Sov. Math. Dokl.* **36** 212–6

[8] Frenkel E and Mukhin E 2001 Combinatorics of q-characters of finite-dimensional representations of quantum affine algebras *Commun. Math. Phys.* **216** 23–57

[9] Frenkel E and Mukhin E 2002 The Hopf algebra Rep $U_q\widehat{\mathfrak{gl}}_\infty$ *Sel. Math. (NS)* **8** 537–635

[10] Humphreys J E 1972 *Introduction to Lie algebras and Representation Theory* (New York: Springer)

[11] Jimbo M 1986 A q-analogue of $U(gl(N+1))$, Hecke algebra and the Yang–Baxter equation *Lett. Math. Phys.* **11** 247–52

[12] Jimbo M 1986 Quantum R-matrix for the generalized Toda system *Commun. Math. Phys.* **102** 537–47

[13] Kashiwara M 2002 On level-zero representation of quantized affine algebras *Duke Math. J.* **112** 117–95

[14] Kitanine N, Maillet J-M and Terras V 1999 Form factors of the XXZ Heisenberg spin-$\frac{1}{2}$ finite chain *Nucl. Phys.* B **554** 647–78

[15] Leclerc B, Nazarov M and Thibon J-Y 2003 Induced representations of affine Hecke algebras and the canonical bases for quantum groups *Studies in Memory of Issai Schur (Progr. in Math.* vol. 210*)* (Basle: Birkhauser) pp 115–53

[16] Maillet J M and Terras V 2000 On the quantum inverse scattering problem *Nucl. Phys.* B **575** 627–44

[17] Molev A I 1994 Gelfand–Tsetlin basis for representations of Yangians *Lett. Math. Phys.* **30** 53–60

[18] Molev A I 1998 Finite-dimensional irreducible representations of twisted Yangians *J. Math. Phys.* **39** 5559–600

[19] Molev A I 2002 Irreducibility criterion for tensor products of Yangian evaluation modules *Duke Math. J.* **112** 307–41

[20] Molev A, Nazarov M and Olshanski G 1996 Yangians and classical Lie algebras *Russ. Math. Surv.* **51** 205–82

[21] Nazarov M and Tarasov V 1994 Yangians and Gelfand–Zetlin bases *Publ. RIMS, Kyoto Univ.* **30** 459–78

[22] Nazarov M and Tarasov V 1998 Representations of Yangians with Gelfand–Zetlin bases *J. Reine Angew. Math.* **496** 181–212

[23] Nazarov M and Tarasov V 1998 On irreducibility of tensor products of Yangian modules *Int. Math. Res. Not.* 125–50

[24] Nazarov M and Tarasov V 2002 On irreducibility of tensor products of Yangian modules associated with skew Young diagrams *Duke Math. J.* **112** 343–78

[25] Reshetikhin N Yu, Takhtajan L A and Faddeev L D 1990 Quantization of Lie groups and Lie algebras *Leningr. Math. J.* 1 193–225

[26] Tolstoy V N 1990 Extremal projectors for quantized Kac-Moody superalgebras and some of their applications *Quantum Groups (Clausthal, 1989) (Lect. Notes Phys.* vol. 370*)* (Berlin: Springer) 118–25

[27] Ueno K, Takebayashi T and Shibukawa Y 1989 Gelfand–Zetlin basis for $U_q(gl(N+1))$-modules *Lett. Math. Phys.* **18** 215–21

[28] Varagnolo M 2000 Quiver varieties and Yangians *Lett. Math. Phys.* **53** 273–83

[29] Varagnolo M and Vasserot E 2002 Standard modules of quantum affine algebras *Duke Math. J.* **111** 509–33